ON SELF-INDUCED OSCILLATIONS OF HEAVY FLUIDS IN TUBES

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Self-induced oscillation systems, described by second order partial differential equations with nonlinear boundary conditions, have been considered in [1-5].

Below we consider self-induced oscillations, which under certain conditions may arise in the motion of high-density fluids in a tube. The problem reduces to the solution of first order quasilinear partial differential equations with non-periodic boundary conditions.

1. The differential equations describing the unsteady motion of a viscous compressible fluid in tubes in terms of a hydraulic resistance are [6]

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \varphi(u) \qquad (1.1)$$

$$\frac{\partial \rho S}{\partial t} + \frac{\partial \rho S u}{\partial x} = 0 \qquad (1.2)$$

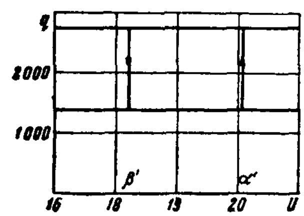

Fig. 1.

Here u is the average velocity of the fluid particles over the cross-section of the tube, p the pressure, ρ the density, and S the cross-sectional area of the tube. To equations (1.1) and (1.2) must be added the equation of state and the condition of isentropy; the function $\varphi(u)$ is taken to be linear for laminar flow and quadratic for turbulent flow [6].

We shall consider the pipe flow to occupy the semi-infinite space

$x \geqslant 0$. If at the end $x = 0$ the pipe is connected to any machine capable of changing the liquid flow (piston pump, valve, turbine, compressor, etc.), and this machine is isolated from the pipe by a chamber used either to regulate the flow or to damp pressure fluctuations (air chamber, equalizing tank, compressor surge tank), then the boundary condition at $x = 0$ will be

$$-u + h\frac{\partial u}{\partial x} = \eta(t) \tag{1.3}$$

where h is a positive constant characterising the type of the chamber, $\eta(t)$ is a known function of time proportional to the flow volume q [6]. We shall consider the flow q to be a function of u of a "relay" type, having segments of multiple-valuedness (Fig. 1). Then condition (1.3) assumes the form

$$\frac{\partial u(0, t)}{\partial x} = \frac{q(u, \partial u/\partial t) + u}{h} \tag{1.4}$$

where q equals either q_1 or q_2 according to Fig. 1.

Assuming that the fluid density ρ is high and therefore the term $\rho^{-1}\partial p/\partial x$ may be neglected in equation (1.1), we shall seek periodic solutions to our system with boundary conditions (1.4).

The main difficulty lies in integrating equation (1.1) with the term $\rho^{-1}\partial p/\partial x$ omitted, under the boundary condition (1.4). Once the velocity $u(x, t)$ is found from that, the remaining quantities ρ, S, p, T are determined from the linear differential equation (1.2) and known relations.

Let us consider a more general problem: to find the periodic solutions to the equation

$$\frac{\partial u}{\partial t} + g'(u)\frac{\partial u}{\partial x} = \varphi(u) \tag{1.5}$$

with the boundary condition

$$\frac{\partial u(0, t)}{\partial x} = Q\left[u(0, t), \frac{\partial u(0, t)}{\partial t}\right] \tag{1.6}$$

The function $Q(z, y)$ will be assumed to be equal to $Q_1(z)$ for $\beta' < z < \alpha$, and to $Q_2(z)$ for $\alpha' < z < \beta$ or for $\beta < z < \alpha'$; moreover, if $y < \gamma$, $Q(z, y) = Q_1(z)$, and as soon as z reaches the value β', the function Q becomes equal to $Q_2(z)$, i.e. there is a jump from the point $z = \beta'$, $Q = Q_1(\beta')$ to the point $z = \beta$, $Q = Q_2(\beta)$. Similarly, for $y > \delta$, $Q = Q_2$, and as soon as z reaches the value α', the function Q becomes equal to $Q_1(z)$, so that there is a jump from the point $z = \alpha'$, $Q = Q_2(\alpha')$ to the point $z = \alpha$, $Q = Q_1(\alpha)$. We assume that $Q_1(z)$ and $Q_2(z)$ are some given monotonic functions, which may be defined also in the exterior of the intervals $\beta' < z < \alpha$ and $\alpha' < z < \beta$ ($\beta < z < \alpha'$) respectively. The intervals $\beta' < z < \alpha$ and $\alpha' < z < \beta$ ($\beta < z < \alpha'$) have a common section.

Since condition (1.6) is not periodic, the desired solution of equation (1.5) must be of the self-induced oscillation type. We shall determine the conditions to be imposed on the functions φ, g, Q_1 and Q_2 to make these solutions possible. We shall assume the functions $\varphi(u)$, $g'(u)$, $Q_1(u)$ and $Q_2(u)$ to be continuous, as well as their derivatives.

2. As is well known [8], a smooth solution to the Cauchy problem of a quasilinear equation of the type (1.5) exists only in a sufficiently small neighborhood of a line on which smooth initial data are given; and if the initial data given on the line are discontinuous, then there exists no smooth solution in general in any neighborhood of the line, however small. Consequently, we should consider generalized solutions of equation (1.5), which are solutions satisfying the conservation law written in integral form

$$\oint_C u\,dx - g(u)\,dt + \iint_G \varphi(u)\,dx\,dt = 0 \tag{2.1}$$

Here G is an arbitrary domain contained in the region of the x, t plane in which the solution is sought, and C is the piecewise smooth boundary of that region. We shall consider generalized solutions of equation (2.1) in a region G contained in the half-plane $x \geqslant 0$.

A solution to (2.1) is a piecewise-continuous function $u(x, t)$, which coincides at all points where it is continuous with a solution to (1.5); the values of $u(x, t)$ on the two sides of the discontinuity surface are connected by the equation

$$\frac{dX}{dt} = \frac{g(u^+) - g(u^-)}{u^+ - u^-} \tag{2.2}$$

Here X denotes the coordinate of the jump discontinuity (or shock); u^+ and u^- are the values of $u(x, t)$ at $x = X(t) + 0$ and $x = X(t) - 0$ respectively. In order that such a discontinuity (or shock) be possible, we must have the condition

$$g'(u^+) < dX/dt < g'(u^-) \tag{2.3}$$

We now find continuous solutions to equation (1.5); its general solution has the form (Φ being an arbitrary function)

$$t = f(u) + \Phi[x - F(u)] \qquad \left(f(u) = \int \frac{du}{\varphi(u)},\ F(u) = \int \frac{g'(u)\,du}{\varphi(u)}\right) \tag{2.4}$$

We shall seek a periodic solution to equation (1.5) with boundary condition (1.6). It is clear from (2.4) that the solution $u(x, t)$ will be periodic with period T if $\Phi^{-1}(z)$ is a periodic function* of period T. The function Φ^{-1} is determined from the boundary condition (1.6). Assuming that the velocity at the entrance of the pipe is known as a function of time, $u(0, t) = u_0(t)$, we may write the solution (2.4) as

$$t = f(u) + u_0^{-1}\{F^{-1}[F(u) - x]\} - f\{F^{-1}[F(u - x)]\} \qquad \textbf{(2.5)}$$

It is clear from (2.5) that the function $\Phi^{-1}(z)$ will be periodic if $u_0(t)$ is periodic with the same period. Using equation (1.5), we now write condition (1.6) in the following fashion:

$$u_0'(t) = \varphi[u_0(t)] - g'[u_0(t)]\, Q\, [u_0(t)] \qquad \textbf{(2.6)}$$

Integrating (2.6), we find

$$t = \int_{u_0(0)}^{u_0} \frac{dz}{\varphi(z) - g'(z)\,Q(z)} \qquad \textbf{(2.7)}$$

The value of the function $Q(z)$ in formula (2.7) is either equal to $Q_1(z)$ or to $Q_2(z)$. We give some initial value $u(0, t_0) = z_0$. To be definite, we assume that $\varphi(z) - g'(z)Q_1(z) < 0$, $g'(u) > 0$ and that $Q = Q_1(z)$ for $z_0 > \beta'$ and $Q = Q_2(z)$ for $z_0 < \beta'$ $(\beta > \beta')$ (since the time t enters equation (1.5) only up to an additive constant, and does not enter (1.6) at all); then we have one of the two equations below for (2.7):

For $z_0 > \beta'$

$$t = \omega^{-1}(u_0) = \int_{\omega(0)}^{u_0} \frac{dz}{\varphi(z) - g'(z)\,Q_1(z)} \qquad \textbf{(2.8)}$$

For $z_0 < \beta'$

$$\left(\begin{matrix} \omega(0) = \alpha, & \Omega(T_1) = \beta \\ \omega(T_1) = \beta', & \Omega(T) = \alpha' \end{matrix}\right)$$

$$t = \Omega^{-1}(u_0) = T_1 + \int_{\Omega(T_1)}^{u_0} \frac{dz}{\varphi(z) - g'(z)\,Q_2(z)} \qquad \textbf{(2.9)}$$

The quantity T_1 will be determined below by formula (2.10).

We first consider the case $\beta' < z_0 < \alpha$. Formula (2.8) holds at $t = T_1$, although the value of u_0, equal to $\omega(t)$, does not equal the value $\omega(T_1)$.

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* Here and hereafter, −1 in the exponent of the characteristic functions $\Phi^{-1}(z)$, $F^{-1}(z)$, $\omega^{-1}(z)$... will denote the inverse functions of $\Phi(z)$, $F(z)$, $\omega(z)$...

At the instant T_1, the solution to (2.7) starts being described by the relation (2.9). As soon as the function $\Omega(t)$ reaches the value $\Omega(T)$, the solution to (2.7) starts to be described by the formula (2.8), in which t is replaced by $t + T$. Thus, the periodic solution to (2.6) is found with the constant T_1 and the period T, according to (2.8) and (2.9), given by the expressions

$$T_1 = \int_{\omega(0)}^{\omega(T_1)} \frac{dz}{\varphi(z) - g'(z)\, Q_1(z)}, \qquad T = T_1 + \int_{\Omega(T_1)}^{\Omega(T)} \frac{dz}{\varphi(z) - g'(z)\, Q_2(z)} \tag{2.10}$$

It is easily seen that for $\alpha' < z_0 < \beta$ or $\beta < z_0 < \alpha'$ the periodic solution to (2.6) may be found in the same manner. If $z_0 > \max(\alpha, \beta)$ or $z_0 < \min(\alpha', \beta')$, then the function $u_0(t)$ given by (2.8) and (2.9) may be periodic only in the limit of $t \to \infty$.

Consequently, a periodic solution to equation (1.5) with non-periodic condition (1.6), for the case $g'(u) > 0$, $\varphi(u) - g'(u)Q_1(u) < 0$ may exist only under the restriction

$$\min(\alpha', \beta') \leqslant u(0, t) \leqslant \max(\alpha, \beta) \tag{2.11}$$

If that solution exists, then at the points at which it is continuous, it is given by formula (2.5), in which the functions $\omega(t)$ and $\Omega(t)$ are periodic with the period T, and $u_0(t)$ is given by

$$u_0(t) = \begin{cases} \omega(t) & \text{for } nT \leqslant t \leqslant nT + T_1 \\ \Omega(t) & \text{for } nT + T_1 \leqslant t \leqslant (n+1)T \end{cases} \quad (n = 0, \pm 1, \ldots) \tag{2.12}$$

while T_1 and T are given by the integral (2.10).

Similar results can be obtained for the remaining cases

$$\begin{array}{ll} g'(u) > 0, & \varphi(u) - g'(u)\, Q_1(u) > 0 \\ g'(u) < 0, & \varphi(u) - g'(u)\, Q_1(u) < 0 \\ g'(u) < 0, & \varphi(u) - g'(u)\, Q_1(u) > 0 \end{array}$$

Let us clarify what restriction must be imposed on the functions $\varphi(u)$ and $g'(u)$, such that condition (2.11) implies the existence of a solution (2.5). If such a solution exists, then a self-induced oscillation occurs in the tube with a period dependent on the functions φ, g, Q_1 and Q_2, which enter the equation (1.5) and the boundary condition (1.6), as can be seen explicitly from formula (2.10).

3. Solution (2.5) to equation (1.5) and boundary condition (2.6) has the form

$$\text{For } nT < \omega^{-1}\{F^{-1}[F(u)-x]\} < nT + T_1 \qquad (n = 0, \pm 1, \ldots) \tag{3.1}$$

$$t = f(u) + \omega^{-1}\{F^{-1}[F(u) - x]\} - f\{F^{-1}[F(u) - x]\}$$

$$\text{For } nT + T_1 < \Omega^{-1}\{F^{-1}[F(u)-x]\} < (n+1)T \qquad (n = 0, \pm 1, \ldots)$$

$$t = f(u) + \Omega^{-1}\{F^{-1}[F(u) - x]\} - f\{F^{-1}[F(u) - x]\} \tag{3.2}$$

(ω^{-1} and Ω^{-1} are multivalued formulas of their arguments)

We note that (3.1) and (3.2) follow from (2.5) if the functions $\omega(t)$, $\Omega(t)$, $F(u)$ are monotonic.

Let us assume that $F'(u) \neq 0$, while the quantities $\varphi(u) - g'(u)Q_i(u)$ ($i = 1, 2$) which have the same signs as the derivatives $\omega'(t)$ and $\Omega'(t)$ respectively (see (2.8) and (2.9)), also nowhere become equal to zero. The characteristics of equation (1.5) are defined by the relationships

$$u = F^{-1}(x - C_2) \tag{3.3}$$

$$t = C_1 + f[F^{-1}(x - C_2)] \tag{3.4}$$

The curves (3.4) are the characteristics in the x, t plane. By virtue of the periodicity of the function $u_0(t)$, the curves (3.4), which lie on the boundaries of the half-strip in which the solution (3.1) is defined, are given by the formulas

$$t = nT - f[\omega(0)] + f\{F^{-1}[x + F(\omega(0))]\} \qquad (n = 0, \pm 1, \ldots) \tag{3.5}$$

$$= nT + T_1 - f[\omega(T_1)] + f\{F^{-1}[x + F(\omega(T_1))]\} \quad (n = 0, \pm 1, \ldots) \tag{3.6}$$

and correspondingly for solution (3.2), by the formulas

$$t = nT + T_1 - f[\Omega(T_1)] + f\{F^{-1}[x + F(\Omega(T_1))]\} \quad (n = 0, \pm 1, \ldots) \tag{3.7}$$

$$t = (n+1)T - f[\Omega(T)] + f\{F^{-1}[x + F(\Omega(T))]\} \quad (n = 0, \pm 1, \ldots) \tag{3.8}$$

Obviously, the half-strips (3.5) to (3.6) and (3.7) to (3.8), which overlap partially, must cover the entire region $x \geqslant 0$ (or else the solution will be non-unique). At the intersection of the half-strips (3.5) to (3.6) and (3.7) to (3.8), the shock occurs. By virtue of the assumed periodicity of the solution, it is sufficient to consider in the x, t plane the half-strip $x \geqslant 0$, $0 \leqslant t \leqslant T$ of width T, or the half-strip between two characteristics (3.5) with $n = 0$ and $n = 1$; in fact, this half-strip consists of pieces constituting the half-strip $0 \leqslant t \leqslant T$.

We first consider the neighborhood of the straight line $x = 0$. A shock, as is well known [11], may be formed either from a discontinuity of the function $u_0(t)$, or from intersection of characteristics. The function $u_0(t)$ has discontinuities at the points $t = nT$, $t = nT + T_1$ ($n = 0, \pm 1, \ldots$). At these points, shocks will originate from $x = 0$, if the following conditions are satisfied:

$$g'[\Omega(T)] < g'[\omega(0)], \qquad g'[\omega(T_1)] < g'[\Omega(T_1)] \tag{3.9}$$

Inequality (3.9) is a consequence of condition (2.3) on the possibility of a shock. Consequently, through the point $O(x = t = 0)$ and the point $C(x = 0,\ t = T_1)$ will pass discontinuity surfaces with initial slopes

$$\frac{dX}{dt} = \frac{g[\omega(0)] - g[\Omega(T)]}{\omega(0) - \Omega(T)}, \qquad \frac{dX}{dt} = \frac{g[\Omega(T_1)] - g[\omega(T_1)]}{\Omega(T_1) - \omega(T_1)} \tag{3.10}$$

respectively. Moreover, the first shock separates the region $u = u^-$ in which the solution (3.1) holds from the region $u = u^+$ in which the solution (3.2) holds; whereas the second separates the region $u = u^-$ in which (3.2) holds from the region $u = u^+$, in which (3.1) holds (Fig. 2).

It is evident that if $g'(u) \neq 0$ ($g''(u) > 0$ or $g''(u) < 0$), and if condition (3.9) holds, then in the neighborhood of the t-axis the characteristics will diverge for $x > 0$, so that the solution (2.1) will exist and be unique for $0 \leqslant x < x_0$ [11].

In the vertically and horizontally shaded regions, the solutions are defined by (3.1) and (3.2) respectively, and the equation of the shocks originating from the points O and C are found from a numerical integration of equation (2.2), in which u^+ and u^- are replaced by the appropriate solution (3.1) or (3.2) (Fig. 2).

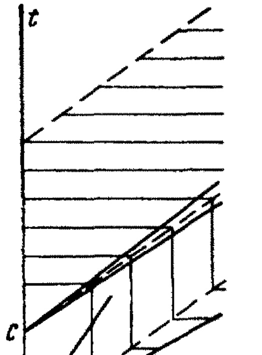

Fig. 2.

Condition (3.9) shows that the constants α, β, α' and β' must satisfy the inequalities

$$g'(\alpha') < g'(\alpha), \qquad g'(\beta') < g'(\beta)$$

if

$$\varphi(z) - g'(z)\, Q_1(z) < 0$$

It is readily seen that the requirement imposed on the functions $g'(u)$ and $\varphi(u)$ to insure that the neighboring characteristics do not intersect for $x > 0$ can be reduced to that already imposed above: $F'(u) \neq 0$, $g'(u) \neq 0$.

It is now clear how to construct the solution to equation (2.1) under the assumptions formulated before on the functions $g'(u)$, $\varphi(u)$ and $Q_i(u)$ up till the point where the two shocks intersect. At the point M where they intersect, where the function $u(x, t)$ has three values, it is necessary to consider the question of the decay of arbitrary discontinuities [9]. Assume, for example, that one of the values u_1 of the function $u(x, t)$ is determined by characteristics from the region in which (3.1) holds, while the other two, u_2 and u_3, from the region where (3.2) holds. Then at the point M the condition (2.3) holds

$$g'(u_2^-) > g'(u_1^+) = g'(u_1^-) > g'(u_3^+) \quad (3.11)$$

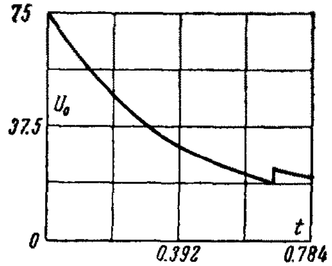

Fig. 3.

Since $u_1^+ = u_1^-$ holds in the interior of the region between the characteristics, in which the value of $u(x, t)$ is given by (3.2), then by virtue of (3.11), condition (2.3) holds for the functions u_2^- and u_3^+: $g'(u_2^-) > g'(u_3^+)$, and a shock passes through the point M with velocity

$$\frac{dX}{dt} = \frac{g(u_3^+) - g(u_2^-)}{u_3^+ - u_2^-}$$

From then on the solution depends only on the values (3.2) of the function $u(x, t)$ on the respective characteristics. It remains to study the stability of the periodic solution found.

It is readily shown that equation (1.5) with boundary condition (1.6) has a stationary solution

$$u = F^{-1}[x + F(\vartheta)] \ (\vartheta \text{ is the value of the function } u(x) \text{ at } x = 0) \quad (3.12)$$

if the equation $\varphi(\vartheta) = g'(\vartheta)\,Q(\vartheta)$ has ϑ as a root.

Linearizing equation (1.5) and condition (1.6) about the stationary solution (3.12) we obtain the stability condition $\lambda < 0$, and the condition for the self-excitation of oscillations [10], $\lambda > 0$, where

$$\lambda = g'(\vartheta)\left\{\frac{d}{d\vartheta}\left(\frac{\varphi(\vartheta)}{g'(\vartheta)}\right) - Q'(\vartheta)\right\} \quad (3.13)$$

Similarly, linearizing the equation (1.5) and boundary condition (1.6) about the periodic solution, we obtain the condition of stability $\Lambda < 0$ and instability $\Lambda > 0$ of the periodic solutions, where

$$\Lambda = \int_0^T \left[\frac{1}{\sigma}\frac{\partial\sigma}{\partial u_0} + \sigma Q_{u_0} + f'(u_0)\Phi^{-1''}(\tau)\, Q_{u_0'}\right]\left(\frac{1 - f'(u_0)\, du_0/dt}{f'(u_0) + \sigma Q u_0'}\right) dt$$

$$(\sigma = F'(u_0) - f'(u_0)\,\Phi^{-1'}[t - f(u_0)], \quad \tau = t - f'(u_0))$$

Figures 3 and 4 show the functions $u(0, t)$ and $u(x, t)$, which are solutions of (1.5) with $g'(u) = u$ and $\varphi(u) = -\mu u^2$ and boundary condition (1.4); the numerical values were taken to be

$h = 1540$ cm, $q_1 = 2750$ cm/sec $(u > \beta',\ \partial u/\partial t < \gamma)$
$q_2 = 1375$ cm/sec $(\alpha' < u < \beta,\ \partial u/\partial t > \delta)$
$\mu = 0.7145 \cdot 10^{-2}$ sec^{-1}, $\alpha = -99.5600$, $\beta = -13.5924$
$\alpha' = -11.8482$, $\beta' = -19.8619$ cm/sec, $\gamma = -35.4676$
$\delta = -21.1574$ cm/sec^2

Curves *1*, *2* and *3* correspond to $t = 0.056$, 0.112, 0.728 sec respectively.

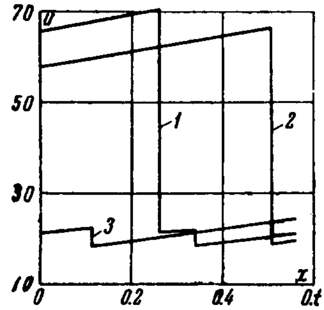

Fig. 4.

4. If the function $g'(u)$ in equation (1.5) is small compared to the constant A, then in some approximations we may replace (1.5) by the equation

$$\frac{\partial u}{\partial t} + A\frac{\partial u}{\partial x} = \varphi(u) \tag{4.1}$$

Using (3.1) and (3.2), we obtain explicit expressions for the solutions of (4.1)

$$u(x, t) = \begin{cases} f^{-1}\{x/A + f[\omega(t - x/A)]\} & \text{for } nT < t - x/A < nT + T_1 \quad (n = 0, \pm 1, \ldots) \\ f^{-1}\{x/A + f[\Omega(t - x/A)]\} & \text{for } nT + T_1 < t - x/A < (n+1)T \end{cases} \tag{4.2}$$

The characteristics (3.4) in this case are straight lines with slope equal to $1/A$; instead of shocks from the points O and C, there will be the characteristics $x = At$ and $x = A(t - T_1)$ respectively, across which the solution may be discontinuous (or its derivative may be discontinuous).

With $\varphi(u) = -\delta u$, the solution (4.2) becomes

$$u(x, t) = \begin{cases} \exp(-\delta x/A)\,\omega(t - x/A) & \text{for } nT < t - x/A < nT + T_1 \\ \exp(-\delta x/A)\,\Omega(t - x/A) & \text{for } nT + T_1 < t - x/A < (n+1)T \end{cases} \tag{4.3}$$

5. We shall now replace equation (1.5) by the equation

$$\frac{\partial u}{\partial t} + A\frac{\partial u}{\partial x} = -\delta u + \nu\frac{\partial^2 u}{\partial x^2} \tag{5.1}$$

and shall study two problems simultaneously.

Problem 1. To find a periodic solution of the linear differential equation (5.1) with the boundary condition (1.6).

Problem 2. To find a periodic solution to equation (5.1) with the boundary condition

$$\frac{\partial u(0, t)}{\partial x} = \frac{\nu}{A}\frac{\partial^2 u}{\partial x^2}(0, t) + Q\left[u(0, t), \frac{\partial u(0,t)}{\partial t}\right] \tag{5.2}$$

The solution of these problems can easily be reduced to integrating the heat equation in the region $z > 0$

$$\frac{\partial w}{\partial \tau} = \frac{\partial^2 w}{\partial z^2} \quad \left|\; \begin{pmatrix} t = \tau T/2\pi, \; x = \vartheta^{-1}z, & \vartheta^2 = 2\pi/\nu T \\ u = w\exp[-(\delta+\nu a^2)t + ax], & a = A/2\nu \end{pmatrix}\right. \tag{5.3}$$

under the boundary conditions (1.6) and (5.2) respectively. From the periodicity of $u(z, \tau)$ in τ with period 2π, we get the functional equation

$$w(z, \tau + 2\pi) = w(z, \tau)\exp(-2\pi\zeta) \qquad (\zeta = -(\delta + \nu a^2)T/2\pi) \tag{5.4}$$

The solution of the heat equation (5.3) satisfying the functional equation (5.4) can be found in [12]; it is given by formulas (2.1) and (2.2) in [12], in this case with $\zeta < 0$. In order that the function $u(z, \tau)$ be bounded between the constants A_0 and B_0 as $z \to \infty$, (these constants appearing in (2.2) of [12]), the following relationship must hold:

$$A_0 + B_0/\sqrt{-\zeta} = 0$$

Using the cited formulas (2.1) and (2.2), we get the expression

$$u(x, t) = \tfrac{1}{2}A_0\exp\{[a - (a^2 + \delta/\nu)^{1/2}]x\} + \tag{5.5}$$
$$+\sum_{k=1}^{\infty}\exp\{[a + \vartheta\rho_k]x\}\{A_k\cos[\vartheta^2 kt - \vartheta\omega_k x] + B_k\sin[\vartheta^2 kt - \vartheta\omega_k x]\}$$

It remains to determine the Fourier coefficients A_k and B_k of the function $u(0, t)$ in (5.5), from the boundary condition (1.6) or (5.2).

Considering the first problem, we use (5.5) to write (1.6) as

$$Q_i\left[\frac{A_0}{2}+\sum_{k=1}^{\infty}(A_k\cos k\tau+B_k\sin k\tau)\right]=v(\tau) \tag{5.6}$$

where

$$v=\frac{A_0}{2}\left[a-\left(a^2+\frac{\delta}{\nu}\right)\right]^{1/2}+\sum_{k=1}^{\infty}\{(aA_k+\vartheta a_k)\cos k\tau+(aB_k+\vartheta b_k)\sin k\tau\}$$

$$(a_k=\rho_k A_k-\omega_k B_k,\ b_k=\rho_k B_k+\omega_k A_k) \tag{5.7}$$

$$Q(z,y)=\begin{cases}Q_1(z) & \text{for } \beta'<z<\alpha,\ y<\gamma\\ Q_2(z) & \text{for } \alpha'<z<\beta \text{ or for } \beta<z<\alpha',\ y>\delta\end{cases} \tag{5.8}$$

If the function (5.8) has the form (1.4)

$$Q_1=q_1+hz, \qquad Q_2=q_2+hz$$

then, using (5.6) to (5.8), we get the coefficients A_k and B_k

$$A_0=\frac{q_1\tau_1+q_2(2\pi-\tau_1)}{\pi[a-\vartheta\sqrt{-\zeta}-h]},\quad \Delta_k=(a+\vartheta\rho_k-h)^2+\vartheta^2\omega_k^2$$

$$B_k=\frac{(q_1-q_2)}{\pi k\Delta_k}[(a+\vartheta\rho_k-h)(1-\cos k\tau_1)-\vartheta\omega_k\sin k\tau_1]$$

$$A_k=\frac{(q_1-q_2)}{\pi k\Delta_k}[(a+\vartheta\rho_k-h)\sin k\tau_1+\vartheta\omega_k(1-\cos k\tau_1)]$$

The quantities τ_1 and T are determined for $q_1>q_2$ as the smallest roots of the transcendental equations

$$\frac{A_0}{2}+\sum_{k=1}^{\infty}A_k=\frac{\alpha+\alpha'}{2}, \qquad \frac{A_0}{2}+\sum_{k=1}^{\infty}(A_k\cos k\tau_1+B_k\sin k\tau_1)=\frac{\beta+\beta'}{2}$$

in which $\alpha'=\alpha$, $\beta'=\beta$, $\gamma=\delta=0$ for this case.

In the more general case, we may consider the inverse problem, considering the function $u_0(\tau)$ to have a form similar to (2.12). Expending the function $u_0(\tau)$ as a Fourier series, we find the coefficients A_k and B_k and consequently, we also determine the form of the functions $Q_1(z)$ and $Q_2(z)$.

Now let us return to the solution of Problem 2. Condition (5.2) is no different from condition (2.6) when $\varphi(u)=-\delta u$ and $g'(u)=A$. Thus, $u(0,\tau)$, T_1 and T are found from the known function Q in exactly the same way as was shown in Section 2. The solution of the problem is given by expression (5.5), where A_k and B_k are the Fourier coefficients of the desired function $u(0, \tau)$, as given by formula (2.6).

We now pass to the limit of ν equal to zero. It is easily seen that solution (5.5) now assumes the form

$$u(x,t) = \exp\left(-\frac{\delta x}{A}\right)\left[\frac{1}{2}A_0 + \sum_{k=1}^{\infty} A_k \cos \nu\theta^2 k\left(t - \frac{x}{A}\right) + B_k \sin \nu\theta^2 k\left(t - \frac{x}{A}\right)\right] \qquad (5.9)$$

while condition (5.2) becomes (1.6). Thus, in the limit of $\nu \to 0$, the solution of both problems coincide with (4.3), which is the solution to equation (4.1) with $\varphi(u) = -\delta u$ under boundary condition (1.6).

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